

# On a class of projectively flat Finsler metrics

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## Abstract

In this paper, we study a class of Finsler metrics composed by a Riemann metric  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  and a 1-form  $\beta = b_i(x)y^i$  called general  $(\alpha, \beta)$ -metrics. We classify those projectively flat when  $\alpha$  is projectively flat. By solving the corresponding nonlinear PDEs, the metrics in this class are totally determined. Then a new group of projectively flat Finsler metrics is found.

**Keywords:** Finsler metric; projectively flat; general  $(\alpha, \beta)$ -metric.

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## 1 Introduction

The regular case of the famous Hilbert's Fourth Problem is to characterize Finsler metrics on an open subset in  $R^n$  whose geodesics are straight lines as a point set. Such Finsler metrics are called *projectively flat* Finsler metrics. In the past years, many non-trivial(non-Minkowskian) projectively flat Finsler metrics have been found. The simplest class is projectively flat Riemannian metrics. As we known, they are equivalent to Riemannian metrics with constant sectional curvature by Beltrami's theorem. However, it is not true in the non-Riemannian case.

Recently, in [6] the first author studied dually flat Finsler metrics arisen from information geometry and found that many dually flat Finsler metrics can be constructed by projectively flat Finsler metrics. More information of dually flat Finsler metrics can be found in [5][9][13]. These also motivate us to search more projectively flat Finsler metrics. Many known examples are related to Riemannian metrics, such as the famous Funk metric  $\Theta = \Theta(x, y)$  found in 1904 and the Berwald's metric  $B = B(x, y)$  found in 1929. This leads to the study of  $(\alpha, \beta)$ -metric defined by a Riemannian metric  $\alpha$  and a 1-form  $\beta$ ,

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \quad (1.1)$$

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where  $\phi = \phi(s)$  satisfies certain condition such that  $F$  is a (positive definite) Finsler metric. The second author gave the equivalent conditions of projectively flat  $(\alpha, \beta)$ -metrics in [12]. Later on, we classified those with constant flag curvature into three types in [8]. The classification of projectively flat Finsler metrics with constant flag curvature were given in [7] [11]. Thus a natural problem is how to find more non-trivial projectively flat Finsler metrics which are not  $(\alpha, \beta)$ -metrics?

A more general class named *general  $(\alpha, \beta)$ -metric* was first introduced by C. Yu and H. Zhu in [15] in the following form.

$$F = \alpha\phi(b^2, \frac{\beta}{\alpha}), \quad (1.2)$$

where  $\alpha$  is a Riemannian metric,  $\beta$  is a 1-form,  $b := \|\beta_x\|_\alpha$  and  $\phi = \phi(b^2, s)$  is a smooth function. It is easy to see that if  $\phi_1 = 0$ , then  $F$  is just a  $(\alpha, \beta)$ -metric. In [15], C. Yu and H. Zhu found a class of projectively flat Finsler metrics in this class with the following three conditions: i)  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  is projectively flat; (ii)  $\beta$  is closed and conformal with respect to  $\alpha$ , i.e., the covariant derivatives of  $\beta$  with respect to  $\alpha$  is  $b_{i|j} = c(x)a_{ij}$ ; (iii)  $\phi = \phi(b^2, s)$  satisfies  $\phi_{22} = 2(\phi_1 - s\phi_{12})$ .

In this paper, we only need the condition that  $\alpha$  is projectively flat and determine all the projectively flat general  $(\alpha, \beta)$ -metrics. Firstly, we get the following main theorem.

**Theorem 1.1** *Let  $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$  be a non-Riemannian Finsler metric on an  $n$ -dimensional manifold with  $n \geq 3$ . If  $\alpha$  is a projectively flat Riemann metric and  $\phi_1 \neq 0$ , then  $F$  is projectively flat if and only if one of the following conditions holds*

- (i)  $\beta$  is parallel with respect to  $\alpha$ ;
- (ii) There are two scalar functions  $c = c(b^2)$  and  $k = k(x)$  such that

$$[cb^2 - (c-1)s^2]\phi_{22} = 2b^2(\phi_1 - s\phi_{12}) \quad (1.3)$$

and

$$b_{i|j} = kc(b^2 a_{ij} - b_i b_j) + kb_i b_j. \quad (1.4)$$

In this case,

$$G^i = {}^\alpha G^i + k\alpha \left\{ (c-1) \frac{(b^2 - s^2)\phi_2}{2\phi} + \frac{b^2(2s\phi_1 + \phi_2)}{2\phi} \right\} y^i. \quad (1.5)$$

Obviously, when  $c = 1$ , it is just the case studied in [15]. By the above theorem, we can see that there are many choices of the function  $c = c(b^2)$  and  $k = k(x)$ . Actually, we find the general solutions of (1.3) and (1.4). Then we immediately obtain the following theorem.

**Theorem 1.2** *Let  $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$  be a non-Riemannian Finsler metric on an  $n$ -dimensional manifold with  $n \geq 3$  such that  $\phi_1 \neq 0$ . Suppose that  $\alpha$  is a*

projectively flat Riemann metric with constant sectional curvature  $\kappa$ , i.e.

$$\alpha_\kappa = \frac{\sqrt{(1 + \kappa|x|^2)|y|^2 - \kappa\langle x, y \rangle^2}}{1 + \kappa|x|^2}.$$

If  $\beta$  is not parallel with respect to  $\alpha_\kappa$ , then  $F$  is projectively flat if and only if there exists a scalar function  $c = c(b^2)$  such that  $\phi$  and  $\beta$  satisfy (1.6) and (1.7) respectively. That is

$$\phi(b^2, s) = f(\mu + \nu s^2) - 2\nu s \int_0^s f'(\mu + \nu z^2) dz + g(b^2)s, \quad (1.6)$$

where  $f = f(t)$  and  $g = g(t)$  are two arbitrary  $\mathcal{C}^\infty$  functions, and

$$\mu = - \int c \nu d(b^2), \quad \nu = -e^{\int \frac{c-1}{b^2} d(b^2)}.$$

$$\beta = e^{-\int \frac{c-1}{2b^2} d(b^2)} \frac{\varepsilon\langle x, y \rangle + (1 + \kappa|x|^2)\langle a, y \rangle - \kappa\langle a, x \rangle\langle x, y \rangle}{(1 + \kappa|x|^2)^{\frac{3}{2}}}. \quad (1.7)$$

In this case,

$$G^i = {}^\alpha G^i + e^{-\int \frac{c-1}{2b^2} d(b^2)} \frac{\varepsilon - \kappa\langle a, x \rangle}{cb^2 \sqrt{1 + \kappa|x|^2}} \alpha \left\{ (c-1) \frac{(b^2 - s^2)\phi_2}{2\phi} + \frac{b^2(2s\phi_1 + \phi_2)}{2\phi} \right\} y^i.$$

Specially, when  $\alpha$  is an Euclidean metric  $|y|$  and  $\beta = \langle x, y \rangle$ , we get the spherically symmetric metric

$$F = |y|\phi(|x|^2, \frac{\langle x, y \rangle}{|y|}).$$

Obviously, it is a special class of general  $(\alpha, \beta)$ -metric. In this case,  $\alpha = |y|$  is projectively flat and  $b_{i|j} = \delta_{ij}$ . Thus by Theorem 1.1 and 1.2 we get the following corollary which was first proved in [16].

**Corollary 1.3** *Let  $F = |y|\phi(|x|^2, \frac{\langle x, y \rangle}{|y|})$  be a non-Riemannian Finsler metric on an  $n$ -dimensional manifold with  $n \geq 3$ . If  $\phi_1 \neq 0$ , then  $F$  is projectively flat if and only if*

$$\phi_{22} = 2(\phi_1 - s\phi_{12}), \quad (1.8)$$

i.e.

$$\phi(b^2, s) = f(b^2 - s^2) + 2s \int_0^s f'(b^2 - z^2) dz + g(b^2)s,$$

where  $f = f(t)$  and  $g = g(t)$  are two arbitrary  $\mathcal{C}^\infty$  functions.

## 2 Preliminaries

In this section, we give some definitions and lemmas needed in this paper. The geodesics of a Finsler metric  $F = F(x, y)$  on an open domain  $\mathcal{U} \subset R^n$  are determined by the following ODEs:

$$\ddot{x} + 2G^i(x, \dot{x}) = 0,$$

where  $G^i = G^i(x, y)$  are called *geodesic coefficients* given by

$$G^i = \frac{1}{4}g^{il}\left\{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\right\}.$$

A Finsler metric  $F = F(x, y)$  is said to be *projectively flat* if its geodesic coefficients  $G^i$  satisfy

$$G^i = Py^i,$$

where  $P = F_{x^k} y^k / (2F)$  is called the *projective factor* of  $F$ . In this paper, we mainly consider the general  $(\alpha, \beta)$ -metrics.

**Proposition 2.1** ([15]) *Let  $M$  be an  $n$ -dimensional manifold.  $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$  is a Finsler metric on  $M$  for any Riemannian metric  $\alpha$  and 1-form  $\beta$  with  $\|\beta\|_\alpha < b_o$  if and only if  $\phi = \phi(b^2, s)$  is a positive  $C^\infty$  function satisfying*

$$\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0 \quad (2.1)$$

when  $n \geq 3$  or

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0$$

when  $n = 2$ , where  $s$  and  $b$  are arbitrary numbers with  $|s| \leq b < b_o$ .

For simplicity, let

$$\begin{aligned} r_{ij} &:= \frac{1}{2}(b_{i|j} + b_{j|i}), \quad r_j := b^i r_{ij}, \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j = b^i s_{ij}, \\ r_{00} &:= r_{ij} y^i y^j, \quad r_{i0} := r_{ij} y^j, \quad r_0 := r_i y^i, \quad s_{i0} := s_{ij} y^j, \quad s_0 = s_i y^i. \end{aligned}$$

In [15], the geodesic coefficients  $G^i$  of a general  $(\alpha, \beta)$ -metric  $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$  were given by

$$\begin{aligned} G^i &= {}^\alpha G^i + \alpha Q s^i_0 + \left\{ \Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha \Omega(r_0 + s_0) \right\} \frac{y^i}{\alpha} \\ &\quad + \left\{ \Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha \Pi(r_0 + s_0) \right\} b^i - \alpha^2 R(r^i + s^i), \end{aligned} \quad (2.2)$$

where  ${}^\alpha G^i$  are geodesic coefficients of  $\alpha$ ,

$$\begin{aligned} Q &= \frac{\phi_2}{\phi - s\phi_2}, \quad R = \frac{\phi_1}{\phi - s\phi_2}, \\ \Theta &= \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]}, \quad \Psi = \frac{\phi_2^2}{2[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]}, \\ \Pi &= \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]}, \quad \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi} \Pi. \end{aligned}$$

### 3 Sufficient Conditions

In this section, we show the sufficient conditions for a general  $(\alpha, \beta)$ -metric to be projectively flat. These conditions are also valid in dimension two.

**Lemma 3.1** *Let  $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$  be a Finsler metric on an  $n$ -dimensional manifold with  $n \geq 2$ . Suppose that  $\beta$  satisfies*

$$b_{i|j} = kc(b^2a_{ij} - b_ib_j) + kb_ib_j, \quad (3.1)$$

where  $c = c(b^2)$  and  $k = k(x)$  are two scalar functions. Assume that  $\phi = \phi(b^2, s)$  satisfies the following PDE:

$$(cb^2 - (c-1)s^2)\phi_{22} = 2b^2(\phi_1 - s\phi_{12}). \quad (3.2)$$

Then the geodesic coefficients  $G^i = G^i(x, y)$  of  $F$  are given by

$$G^i = {}^\alpha G^i + k\alpha \left\{ (c-1) \frac{(b^2 - s^2)\phi_2}{2\phi} + \frac{b^2(2s\phi_1 + \phi_2)}{2\phi} \right\} y^i.$$

**Proof:** By the assumption, we have

$$r_{00} = kc(b^2\alpha^2 - \beta^2) + k\beta^2, \quad r_0 = kb^2\beta, \quad r^i = kb^2b^i, \\ r = kb^4, \quad s_0 = 0, \quad s^i_0 = 0.$$

Substituting them into (2.2) yields

$$G^i = {}^\alpha G^i + k\alpha \{ \Theta[c(b^2 - s^2) + s^2 + 2b^4R] + b^2s\Omega \} y^i \\ + k\alpha^2 \{ \Psi[c(b^2 - s^2) + s^2 + 2b^4R] + b^2s\Pi - b^2R \} b^i.$$

By substituting the expression of  $\Theta, \Psi, R, \Omega$  and  $\Pi$  into the above equation we get

$$G^i = {}^\alpha G^i + k\alpha \left\{ \frac{(b^2 - s^2)(s\phi_{22} + \phi_2)}{2\phi} (c-1) + \frac{(s\phi_{22} + 2s^2\phi_{12} + \phi_2)b^2}{2\phi} \right. \\ \left. + \frac{b^2(2s\phi - 2s^2\phi_2 + b^2\phi_2 + b^2s\phi_{22} - s^3\phi_{22})}{\phi(\phi - s\phi_2 + b^2\phi_{22} - s^2\phi_{22})} [\phi_1 - s\phi_{12} - \frac{c(b^2 - s^2) + s^2}{2b^2}\phi_{22}] \right\} y^i \\ + k\alpha^2 \frac{(cb^2 - (c-1)s^2)\phi_{22} - 2b^2(\phi_1 - s\phi_{12})}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})} b^i.$$

Then by (3.2), we obtain

$$G^i = {}^\alpha G^i + k\alpha \left\{ \frac{(b^2 - s^2)\phi_2}{2\phi} (c-1) + \frac{b^2(2s\phi_1 + \phi_2)}{2\phi} \right\} y^i.$$

Q.E.D.

**Proof of the sufficiency of Theorem 1.1:** By the assumption,  $\alpha$  is projectively flat, then  ${}^\alpha G^i = {}^\alpha P y^i$ , where  ${}^\alpha P$  is projective factor of  $\alpha$ . Thus  $F$  is a projectively flat Finsler metric by Lemma 3.1. Q.E.D.

## 4 Necessary Conditions

The necessity of Theorem 1.1 can be obtained by the following lemma.

**Lemma 4.1** *Let  $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$  be a non-Riemannian Finsler metric on an  $n$ -dimensional manifold with  $n \geq 3$  and  $\phi_1 \neq 0$ . Assume that  $\alpha$  is projectively flat and  $\beta$  is not parallel with respect to  $\alpha$ . If  $F$  is projectively flat, then there are two scalar functions  $c = c(b^2)$  and  $k = k(x)$  such that*

$$(cb^2 - (c-1)s^2)\phi_{22} = 2b^2(\phi_1 - s\phi_{12}) \quad (4.1)$$

and

$$b_{i|j} = kc(b^2a_{ij} - b_ib_j) + kb_ib_j. \quad (4.2)$$

**Proof:** By the assumption,  $\alpha$  is projectively flat, then  ${}^\alpha G^i = {}^\alpha P y^i$ , where  ${}^\alpha P$  is the projective factor of  $\alpha$ . If  $F$  is projectively flat, then by (2.2) there exists  $\bar{P} = \bar{P}(x, y)$  such that

$$\alpha Q s^i_0 + \{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha \Pi(r_0 + s_0)\} b^i - \alpha^2 R(r^i + s^i) = \bar{P} y^i. \quad (4.3)$$

Contracting the above equation with  $y_i = a_{ij}y^j$ , we have

$$\bar{P} = \frac{s}{\alpha} \{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha \Pi(r_0 + s_0)\} - R(r_0 + s_0). \quad (4.4)$$

Substituting it back into (4.3) yields

$$\begin{aligned} & \{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha \Pi(r_0 + s_0)\} (b^i - \frac{s}{\alpha} y^i) \\ & + \alpha Q s^i_0 - R\{\alpha^2(r^i + s^i) + (r_0 + s_0)y^i\} = 0. \end{aligned} \quad (4.5)$$

Since the dependence of  $\phi$  on  $b^2$  and  $s$  is unclear, it is difficult to solve the above equation directly. To overcome this problem, we choose a special coordinate system at a point as in [8]. Fix an arbitrary point  $x_o \in \mathcal{U} \subset R^n$ . Make a change of coordinates:  $(s, y^a) \rightarrow (y^i)$  by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad y^a = y^a, \quad a = 2, \dots, n$$

where  $\bar{\alpha} := \sqrt{\sum_{a=2}^n (y^a)^2}$ . Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}.$$

and

$$r_{00} = r_{11} \frac{s^2}{b^2 - s^2} \bar{\alpha}^2 + 2\bar{r}_{10} \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} + \bar{r}_{00},$$

$$r_0 = \frac{bsr_{11}}{\sqrt{b^2 - s^2}} \bar{\alpha} + b\bar{r}_{10}, \quad s_0 = \bar{s}_0,$$

$$s^1_0 = \bar{s}^1_0, \quad s^a_0 = \frac{s\bar{s}^a_1}{\sqrt{b^2 - s^2}}\bar{\alpha} + \bar{s}^a_0,$$

where  $\bar{r}_{10} = r_{1a}y^a$ ,  $\bar{s}_0 = s_ay^a$ ,  $\bar{s}^1_0 = s_{1a}y^a$  and  $\bar{s}^a_0 = s^a_by^b$ . By a direct computation (4.5) is equivalent to

$$(b^2 - s^2)\Psi\bar{r}_{00} + \left\{ b^2[(b^2 - s^2)(\Pi - 2Q\Psi) + Q + sR]\bar{s}_{10} + [(b^2 - s^2)(b^2\Pi + 2s\Psi) + b^2sR]\bar{r}_{10} \right\} \frac{\bar{\alpha}}{\sqrt{b^2 - s^2}} + r_{11}(2b^4R\Psi + s^2\Psi + b^2s\Pi - b^2R)\bar{\alpha}^2 = 0, \quad (4.6)$$

and

$$\begin{aligned} & s\Psi(b^2 - s^2)y_a\bar{r}_{00} + \sqrt{b^2 - s^2} \left\{ b^2(-2sQ\Psi - R + s\Pi)\bar{s}_{10} \right. \\ & \quad \left. + (b^2s\Pi + 2s^2\Psi - b^2R)\bar{r}_{10} \right\} y_a\bar{\alpha} \\ & \quad - \left\{ b^2\bar{s}_{a0}Q - y_asr_{11}(2b^4R\Psi + s^2\Psi + b^2s\Pi - b^2R) \right\} \bar{\alpha}^2 \\ & \quad + \left\{ \frac{b^2}{\sqrt{b^2 - s^2}} [b^2R(r_{1a} + s_{1a}) + sQs_{1a}] \right\} \bar{\alpha}^3 = 0. \end{aligned} \quad (4.7)$$

By  $\bar{\alpha} = \sqrt{\sum_{a=2}^n (y^a)^2}$ , (4.6) is equivalent to

$$(b^2 - s^2)\Psi\bar{r}_{00} + r_{11}(2b^4R\Psi + s^2\Psi + b^2s\Pi - b^2R)\bar{\alpha}^2 = 0 \quad (4.8)$$

and

$$b^2[(b^2 - s^2)(\Pi - 2Q\Psi) + Q + sR]\bar{s}_{10} + [(b^2 - s^2)(b^2\Pi + 2s\Psi) + b^2sR]\bar{r}_{10} = 0. \quad (4.9)$$

(4.7) is equivalent to

$$s\Psi(b^2 - s^2)y_a\bar{r}_{00} - \left\{ b^2Q\bar{s}_{a0} - y_asr_{11}(2b^4R\Psi + s^2\Psi + b^2s\Pi - b^2R) \right\} \bar{\alpha}^2 = 0 \quad (4.10)$$

and

$$\begin{aligned} & \sqrt{b^2 - s^2} \left\{ b^2(-2sQ\Psi - R + s\Pi)\bar{s}_{10} + (b^2s\Pi + 2s^2\Psi - b^2R)\bar{r}_{10} \right\} y_a \\ & \quad + \frac{b^2}{\sqrt{b^2 - s^2}} [b^2R(r_{1a} + s_{1a}) + sQs_{1a}] \bar{\alpha}^2 = 0. \end{aligned} \quad (4.11)$$

(4.8)  $\times sy_a -$  (4.10) yields

$$b^2Q\bar{s}_{a0}\bar{\alpha}^2 = 0. \quad (4.12)$$

Then by the arbitrary choice of  $x_o$ ,  $Q = 0$  or  $s_{ab} = 0$ . By the assumption that  $F$  is non-Riemannian, we have  $\phi_2 \neq 0$ . Then  $Q \neq 0$ . Thus  $s_{ab} = 0$ .

Differentiating (4.9) with respect to  $y^a$  yields

$$b^2[(b^2 - s^2)(\Pi - 2Q\Psi) + Q + sR]s_{1a} + [(b^2 - s^2)(b^2\Pi + 2s\Psi) + b^2sR]r_{1a} = 0. \quad (4.13)$$

By (4.11) it can be seen that

$$(b^2 R + sQ)s_{1a} + b^2 R r_{1a} = 0 \quad (4.14)$$

and

$$b^2(-2sQ\Psi - R + s\Pi)s_{1a} + (b^2 s\Pi + 2s^2\Psi - b^2 R)r_{1a} = 0 \quad (4.15)$$

because  $\alpha^2$  is indivisible by  $y_a$ . We claim that  $s_{1a} = 0$ . If  $R = 0$  at  $x_0$ , then by (4.14) and  $Q \neq 0$ , we get  $s_{1a} = 0$ . If  $R \neq 0$  and  $s_{1a} \neq 0$  at  $x_0$ , then (4.14) becomes to

$$b^2 + \frac{sQ}{R} + b^2 \frac{r_{1a}}{s_{1a}} = 0.$$

Differentiating the above equation with respect to  $s$  yields

$$\left[ \frac{s\phi_2}{\phi_1} \right]_2 = \frac{\phi_1\phi_2 + s\phi_1\phi_{22} - s\phi_2\phi_{12}}{\phi_1^2} = 0.$$

Then

$$s\phi_1\phi_{22} + \phi_2(\phi_1 - s\phi_{12}) = 0. \quad (4.16)$$

It is easy to see that (4.14) and (4.15) both are linear equations of  $s_{1a}$  and  $r_{1a}$ . If  $s_{1a} \neq 0$ , then

$$\begin{vmatrix} (b^2 R + sQ) & b^2 R \\ b^2(-2sQ\Psi - R + s\Pi) & b^2 s\Pi + 2s^2\Psi - b^2 R \end{vmatrix} = 0.$$

It is equivalent to

$$s(b^2\phi_1 + s\phi_2)\phi_{22} + b^2\phi_2(-\phi_1 + s\phi_{12}) = 0. \quad (4.17)$$

(4.16)  $\times b^2 +$  (4.17) yields

$$s(2b^2\phi_1 + s\phi_2)\phi_{22} = 0. \quad (4.18)$$

Then  $\phi_{22} = 0$  or  $2b^2\phi_1 + s\phi_2 = 0$ . If  $\phi_{22} = 0$ , then by (4.17) we get

$$\phi_2(-\phi_1 + s\phi_{12}) = 0.$$

By the assumption that  $F = \alpha\phi$  is non-Riemannian, we have  $\phi_2 \neq 0$ . Then  $\phi_1 = 0$  when  $s = 0$ . It is excluded. Thus  $\phi_{22} \neq 0$  and

$$2b^2\phi_1 + s\phi_2 = 0.$$

Then  $\phi_1 = 0$  when  $s = 0$ . It is excluded. Thus  $s_{1a} = 0$ .

Substituting  $s_{1a} = 0$  into (4.14) and (4.15) yields

$$b^2 R r_{1a} = 0 \quad (4.19)$$

and

$$(b^2 s\Pi + 2s^2\Psi - b^2 R)r_{1a} = 0. \quad (4.20)$$



If  $R = 0$ , then  $\phi_1 = 0$  and  $\phi_{12} = 0$ . Then (4.20) becomes to

$$2s^2\Psi r_{1a} = 0.$$

By the assumption that  $\phi_{22} \neq 0$  which implies  $\Psi \neq 0$ , then we get

$$r_{1a} = 0.$$

Differentiating (4.8) with respect to  $y^a$  and  $y^b$  yields

$$(b^2 - s^2)\Psi r_{ab} + r_{11}(2b^4 R\Psi + s^2\Psi + b^2 s\Pi - b^2 R)\delta_{ab} = 0.$$

Set  $r_{11} = kb^2$ , ( $k = k(x)$ ). Then there exists a number  $c$  such that

$$r_{ab} = ckb^2\delta_{ab}.$$

Thus we obtain (4.2).

In this case, (4.8) becomes

$$(b^2 - s^2)\Psi ck + (2b^4 R\Psi + s^2\Psi + b^2 s\Pi - b^2 R)k = 0.$$

Substituting the expressions of  $\Psi$ ,  $\Pi$  and  $R$  into the above equation yields

$$k \frac{[cb^2 - (c-1)s^2]\phi_{22} - 2b^2(\phi_1 - s\phi_{12})}{2[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]} = 0. \quad (4.21)$$

If  $k = 0$ , then  $b_{i|j} = 0$ . It is contrary to the assumption. Thus we obtain (4.2).  
Q.E.D.

## 5 General solutions of (1.3)

To determine the projectively flat metrics in Theorem 1.1, the efficient way is to solve (1.3), (1.4). In this section, we first give the general solutions of (1.3), then construct some special explicit solutions.

**Proposition 5.1** *The general solutions of (1.3) are given by*

$$\phi(b^2, s) = f(\mu + \nu s^2) - 2\nu s \int_0^s f'(\mu + \nu z^2) dz + g(b^2)s, \quad (5.1)$$

where  $f = f(t)$  and  $g = g(t)$  are two arbitrary  $\mathcal{C}^\infty$  functions,

$$\mu = - \int c\nu d(b^2), \quad \nu = -e^{\int \frac{c-1}{b^2} d(b^2)}. \quad (5.2)$$

**Proof:** Consider the following variable substitution,

$$u = \mu + \nu s^2, \quad v = s,$$

where  $\mu$  and  $\nu$  are given by (5.2) By the implicit differentiation, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial v} \mu + s^2 \frac{\partial}{\partial v} \nu + 2s\nu \\ &= -c\nu \frac{\partial(b^2)}{\partial v} + s^2 \nu \frac{c-1}{b^2} \frac{\partial(b^2)}{\partial v} + 2s\nu \end{aligned}$$

Then

$$\frac{\partial(b^2)}{\partial v} = \frac{2sb^2}{cb^2 - (c-1)s^2}.$$

By the above equation and (1.3), we get

$$\begin{aligned} \frac{\partial}{\partial v}(\phi - s\phi_2) &= (\phi_1 - s\phi_{12}) \frac{\partial(b^2)}{\partial v} - s\phi_{22} \\ &= (\phi_1 - s\phi_{12}) \frac{2sb^2}{cb^2 - (c-1)s^2} - s\phi_{22} \\ &= 0. \end{aligned} \tag{5.3}$$

This means that

$$\phi - s\phi_2 = f(\mu + \nu s^2) \tag{5.4}$$

is a function of  $\mu + \nu s^2$ . Differentiating (5.4) with respect to  $s$  yields

$$\phi_{22} = 2\nu f'(\mu + \nu s^2).$$

Thus

$$\phi_2 = 2\nu \int_0^s f'(\mu + \nu z^2) dz + g(b^2).$$

Plugging it into (5.4) gives (5.1).

Q.E.D.

Let  $c = \lambda = \text{const.}$ , then (1.3) becomes

$$(\lambda b^2 - (\lambda - 1)s^2)\phi_{22} = 2b^2(\phi_1 - s\phi_{12}). \tag{5.5}$$

By Proposition 5.1, we immediately have

**Lemma 5.2** *The general solutions of (5.5) are given by*

$$\phi(b^2, s) = f(b^{2\lambda} - b^{2(\lambda-1)}s^2) + 2b^{2(\lambda-1)}s \int_0^s f'(b^{2\lambda} - b^{2(\lambda-1)}z^2) dz + g(b^2)s,$$

where  $f = f(t)$  and  $g = g(t)$  are two arbitrary  $\mathcal{C}^\infty$  functions.

In this case,

$$\phi - s\phi_2 = f(t), \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} = f(t) + 2tf'(t),$$

where  $t = b^{2(\lambda-1)}(b^2 - s^2)$ . Let  $f(0) > 0$  and  $f'(t) \geq 0$ , then  $\phi$  satisfies (2.1). Thus  $F = \alpha\phi$  is a Finsler metric. Then we can construct many projectively

flat general  $(\alpha, \beta)$ -metrics by choosing special  $f$  and  $g$ . The following are some special solutions.

(i)  $f = 1$ ,

$$\phi = 1 + g(b^2)s.$$

In this case,  $F = \alpha\phi(b^2, s)$  is a Randers metric.

(ii)  $f = \frac{1}{\sqrt{1-t}}$ ,

$$\phi = \frac{\sqrt{1 - b^{2\lambda} + b^{2(\lambda-1)}s^2}}{1 - b^{2\lambda}} + g(b^2)s.$$

In this case,  $F = \alpha\phi(b^2, s)$  is also a Randers metric.

(iii)  $f = 1 + t$ ,

$$\phi = b^{2(\lambda-1)}s^2 + g(b^{2\lambda})s + 1 + b^{2\lambda}.$$

(iv)  $f = 1 + t^2$ ,

$$\phi = -\frac{b^{4(\lambda-1)}}{3}s^4 + 2b^{4\lambda-2}s^2 + g(b^2)s + 1 + b^{4\lambda}.$$

(v)  $f = \ln(1 + t)$ ,

$$\phi = g(b^2)s + \frac{2b^{\lambda-1} \tanh^{-1} \left[ \frac{b^{\lambda-1}s}{\sqrt{1+b^{2\lambda}}} \right]}{\sqrt{1+b^{2\lambda}}}s + \ln(1 + b^{2\lambda} - b^{2(\lambda-1)}s^2)$$

It is easy to see that (iii), (iv) and (v) are all new solutions.

## 6 General solutions of (1.4)

In this section we solve (1.4) in Theorem 1.1 when  $\alpha$  is a projectively flat Riemannian metric. By Beltrami's theorem, projectively flat Riemannian metrics are equivalent to Riemannian metrics with constant sectional curvature. Let  $\alpha_\kappa$  be a Riemannian metric with constant sectional curvature  $\kappa$ , then there is a local coordinate system such that

$$\alpha_\kappa = \frac{\sqrt{(1 + \kappa|x|^2)|y|^2 - \kappa\langle x, y \rangle^2}}{1 + \kappa|x|^2}.$$

Our method is to take a deformation of  $\beta$  first, then find a conformal 1-form with respect to  $\alpha_\kappa$ . The following lemma can be obtained by a direct computation.

**Lemma 6.1** *Let  $\tilde{\beta} = \rho(b^2)\beta$ , then*

$$\tilde{b}_{i|j} = \rho b_{i|j} + 2\rho' b_i(r_j + s_j).$$

Let  $\rho = e^{\int \frac{c-1}{2b^2} d(b^2)}$ , then by (1.4) and the above lemma we have

$$\tilde{b}_{i|j} = ckb^2 \rho a_{ij}.$$

Then by the result in [14], we have

$$\tilde{\beta} = \frac{\varepsilon \langle x, y \rangle + (1 + \kappa |x|^2) \langle a, y \rangle - \kappa \langle a, x \rangle \langle x, y \rangle}{(1 + \kappa |x|^2)^{\frac{3}{2}}},$$

$$\tilde{b}_{i|j} = \frac{\varepsilon - \kappa \langle a, x \rangle}{\sqrt{1 + \kappa |x|^2}} a_{ij},$$

where  $\varepsilon$  is a constant number and  $a$  is a constant vector. When  $c = 0$ , then  $\varepsilon = 0$  and  $a = 0$ . In this case,  $\tilde{\beta} = \beta = 0$ . Thus, when  $c \neq 0$ , we obtain

**Proposition 6.2** *If  $\beta$  is a 1-form satisfies (1.4), where  $c \neq 0$  and  $\alpha_\kappa = \sqrt{a_{ij}y^i y^j}$  is a Riemannian metric with constant sectional flag curvature  $\kappa$ , then*

$$\beta = e^{-\int \frac{c-1}{2b^2} d(b^2)} \frac{\varepsilon \langle x, y \rangle + (1 + \kappa |x|^2) \langle a, y \rangle - \kappa \langle a, x \rangle \langle x, y \rangle}{(1 + \kappa |x|^2)^{\frac{3}{2}}}.$$

In this case, the scalar function  $k = k(x)$  in (1.4) is given by

$$k(x) = e^{-\int \frac{c-1}{2b^2} d(b^2)} \frac{\varepsilon - \kappa \langle a, x \rangle}{cb^2 \sqrt{1 + \kappa |x|^2}}.$$

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